Calcolare il seguente integrale indefinito :

$$\int \frac{1}{1+x^4}\, dx$$

Osserviamo che la quantità al denominatore della funzione integranda si può scrivere come segue

$$1+x^4=\left(x^2+1\right)-2\,x^2=\left(x^2+1\right)-\left(\sqrt{2}\;x\right)^2 \text{ ovvero}$$

$$1+x^4=\left(x^2+\sqrt{2}\;x+1\right)\left(x^2-\sqrt{2}\;x+1\right)$$

Posto quindi :

$$\frac{1}{x^4+1}=\frac{\mathrm{Ax}+B}{x^2+\sqrt{2}\;x+1}+\frac{\mathrm{Cx}+D}{x^2-\sqrt{2}\;x+1}$$

Si dovrà avere che

$$1=(\mathrm{Ax}+B)\left(x^2-\sqrt{2}\;x+1\right)+(\mathrm{Cx}+D)\left(x^2+\sqrt{2}\;x+1\right)$$

da cui moltiplicando e raccogliendo i fattori comuni segue che

$$1=\mathrm{Ax}^3-\sqrt{2}\;A\,x^2+\mathrm{Ax}+B+\mathrm{Bx}^2-\\ \sqrt{2}\;\mathrm{Bx}+\sqrt{2}\;C\,x^2+\mathrm{Cx}^3+\mathrm{Cx}+D+\mathrm{Dx}^2+\sqrt{2}\;\mathrm{Dx}$$

$$1=(A+C)\,x^3+(C-A)\sqrt{2}\;x^2+(A+C)\,x+(B+D)\,x^2+\sqrt{2}\;(D-B)\,x+B+D$$

$$1=(A+C)\,x^3+\left[(C-A)\sqrt{2}\;+B+D\right]x^2+\left[A+C+\sqrt{2}\;(D-B)\right]x+B+D$$

dovrà risultare pertanto che

$$\begin{cases} A+C=0 \Longrightarrow A=-C & \square \\ (C-A)\sqrt{2}\;+B+D=0 \\ A+C+\sqrt{2}\;(D-B)=0 \\ B+D=1 \end{cases}$$

ovvero

$$\begin{cases} A = -C \\ (C - A)\sqrt{2} + 1 = 0 \Longrightarrow C - A = -\frac{1}{\sqrt{2}} \\ \sqrt{2}\,(D - B) = 0 \Longrightarrow D = B \\ B + D = 1 \end{cases}$$ □

da cui

$$\begin{cases} C - A = -\frac{1}{\sqrt{2}} \Longrightarrow 2\,C = -\frac{1}{\sqrt{2}} \text{ quindi } C = -\frac{1}{2\sqrt{2}} \text{ ed } A = \frac{1}{2\sqrt{2}} \\ \quad B + D = 1 \text{ ma essendo } B = D \text{ deriva che } B = D = \frac{1}{2} \end{cases}$$

in conclusione si ottiene che

$$A = \frac{1}{2\sqrt{2}}, \quad B = \frac{1}{2}, \quad C = -\frac{1}{2\sqrt{2}}, \quad D = \frac{1}{2}$$

Pertanto possiamo affermare che

$$\frac{1}{1+x^4} = \frac{\frac{1}{2\sqrt{2}}\,x + \frac{1}{2}}{x^2 + \sqrt{2}\,x + 1} + \frac{-\frac{1}{2\sqrt{2}}\,x + \frac{1}{2}}{x^2 - \sqrt{2}\,x + 1}$$

da cui

$$\frac{1}{1+x^4} = \frac{\frac{1}{2\sqrt{2}}\,x + \frac{1}{2}}{x^2 + \sqrt{2}\,x + 1} - \frac{\frac{1}{2\sqrt{2}}\,x - \frac{1}{2}}{x^2 - \sqrt{2}\,x + 1} =$$

$$\frac{1}{2\sqrt{2}}\,\frac{1}{2}\,\frac{2\,x}{x^2 + \sqrt{2}\,x + 1} + \frac{1}{2}\,\frac{1}{x^2 + \sqrt{2}\,x + 1} -$$

$$\frac{1}{2\sqrt{2}}\,\frac{1}{2}\,\frac{2\,x}{x^2 - \sqrt{2}\,x + 1} + \frac{1}{2}\,\frac{1}{x^2 - \sqrt{2}\,x + 1} =$$

$$\frac{1}{4\sqrt{2}}\,\frac{2\,x + \sqrt{2} - \sqrt{2}}{x^2 + \sqrt{2}\,x + 1} + \frac{1}{2}\,\frac{1}{x^2 + \sqrt{2}\,x + 1} -$$

$$\frac{1}{4\sqrt{2}}\,\frac{2\,x - \sqrt{2} + \sqrt{2}}{x^2 - \sqrt{2}\,x + 1} + \frac{1}{2}\,\frac{1}{x^2 - \sqrt{2}\,x + 1} =$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x+\sqrt{2}}{x^2+\sqrt{2}\ x+1}+\frac{\frac{1}{2}-\frac{1}{4\sqrt{2}}\sqrt{2}}{x^2+\sqrt{2}\ x+1}-$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x-\sqrt{2}}{x^2-\sqrt{2}\ x+1}+\frac{\frac{1}{2}-\frac{1}{4\sqrt{2}}\sqrt{2}}{x^2-\sqrt{2}\ x+1}=$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x+\sqrt{2}}{x^2+\sqrt{2}\ x+1}+\frac{\frac{1}{2}-\frac{1}{4}}{x^2+\sqrt{2}\ x+1}-$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x-\sqrt{2}}{x^2-\sqrt{2}\ x+1}+\frac{\frac{1}{2}-\frac{1}{4}}{x^2-\sqrt{2}\ x+1}=$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x+\sqrt{2}}{x^2+\sqrt{2}\ x+1}+\frac{\frac{1}{4}}{x^2+\sqrt{2}\ x+1}-$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x-\sqrt{2}}{x^2-\sqrt{2}\ x+1}+\frac{\frac{1}{4}}{x^2-\sqrt{2}\ x+1}=$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x+\sqrt{2}}{x^2+\sqrt{2}\ x+1}+\frac{\frac{1}{4}}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}-$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x-\sqrt{2}}{x^2-\sqrt{2}\ x+1}+\frac{\frac{1}{4}}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}=$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x+\sqrt{2}}{x^2+\sqrt{2}\ x+1}+\frac{\frac{1}{4}}{\frac{1}{2}\left[2\left(\frac{2x+\sqrt{2}}{2}\right)^2+1\right]}-$$

$$\frac{1}{4\sqrt{2}}\ \frac{2x-\sqrt{2}}{x^2-\sqrt{2}\ x+1}+\frac{\frac{1}{4}}{\frac{1}{2}\left[2\left(\frac{2x-\sqrt{2}}{2}\right)^2+1\right]}=$$

$$\frac{1}{4\sqrt{2}}\,\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}+\frac{1}{4}\,2\,\frac{1}{1+\left(\frac{2\sqrt{2}\,x+2}{2}\right)^2}-$$

$$\frac{1}{4\sqrt{2}}\,\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}+\frac{1}{4}\,2\,\frac{1}{1+\left(\frac{2\sqrt{2}\,x-2}{2}\right)^2}=$$

$$\frac{1}{4\sqrt{2}}\,\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}+\frac{1}{2}\,\frac{1}{1+\left(\sqrt{2}\,x+1\right)^2}-$$

$$\frac{1}{4\sqrt{2}}\,\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}+\frac{1}{2}\,\frac{1}{1+\left(\sqrt{2}\,x-1\right)^2}\Longrightarrow$$

$$\frac{1}{x^4+1}=\frac{1}{4\sqrt{2}}\,\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}+$$

$$\frac{1}{2}\,\frac{1}{1+\left(\sqrt{2}\,x+1\right)^2}-\frac{1}{4\sqrt{2}}\,\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}+\frac{1}{2}\,\frac{1}{1+\left(\sqrt{2}\,x-1\right)^2}$$

Integrando membro a membro possiamo scrivere tale relazione

$$\int\frac{1}{x^4+1}\,dx=\frac{1}{4\sqrt{2}}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1}\,dx+\frac{1}{2\sqrt{2}}\int\frac{\sqrt{2}}{1+\left(\sqrt{2}\,x+1\right)^2}\,dx-$$

$$\frac{1}{4\sqrt{2}}\int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}\,dx+\frac{1}{2\sqrt{2}}\int\frac{\sqrt{2}}{1+\left(\sqrt{2}\,x-1\right)^2}\,dx$$

In conclusione sarà

$$\int\frac{1}{x^4+1}\,dx=\frac{1}{4\sqrt{2}}\Big[\log\left(x^2+\sqrt{2}\,x+1\right)-$$

$$\log\left(x^2-\sqrt{2}\,x+1\right)+2\arctan\left(\sqrt{2}\,x-1\right)+2\arctan\left(\sqrt{2}\,x+1\right)\Big]+c$$